

CLT for an iterated integral with respect to fBm with $H > 1/2$

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Abstract

We construct an iterated stochastic integral with fractional Brownian motion with $H > 1/2$. The first integrand is a deterministic function, and each successive integral is with respect to an independent fBm. We show that this symmetric stochastic integral is equal to the Malliavin divergence integral. By a version of the Fourth Moment theorem of Nualart and Peccati [7], we show that a family of such integrals converges in distribution to a scaled Brownian motion. An application is an approximation to the windings for a planar fBm, previously studied by Baudoin and Nualart [2].

1 Introduction

Let $B = \{(B_t^1, \dots, B_t^q), t \geq 0\}$ be a multidimensional fractional Brownian motion (fBm) with Hurst parameter $H > 1/2$. We study the asymptotic behavior as $k \rightarrow \infty$ of multiple stochastic integrals of the particular form:

$$Y_{k^t} := \int_{[0, \infty)^q} s_q^{-qH} \mathbf{1}_{\{1 \leq s_1 < \dots < s_q \leq k^t\}} dB = \int_1^{k^t} \int_1^{s_q} \dots \int_1^{s_2} s_q^{-qH} dB_{s_1}^1 \dots dB_{s_{q-1}}^{q-1} dB_{s_q}^q$$

where $t > 0$ and each iterated integral is a pathwise symmetric integral in the sense of Russo and Vallois [9], and also a divergence integral. Our main result is a central limit theorem for the process $\{Y_{k^t}, t \geq 0\}$, namely that $\frac{Y_{k^t}}{\sqrt{\log k}}$ converges in distribution as $k \rightarrow \infty$ to a scaled Brownian motion. Our approach uses the techniques of Malliavin calculus, where we express Y_{k^t} in terms of the divergence integral δ , which coincides with the multiple Wiener-Itô stochastic integral in this case. In our proof, convergence of finite-dimensional distributions follows from a multi-dimensional version of the Fourth Moment Theorem [7, 8], which gives conditions for weak convergence to a Gaussian random variable (see section 2.4). Functional convergence to a Brownian motion is proved by investigating tightness. In addition to the proof, we are able to comment on the rate of convergence (which is fairly slow: $\sim (\log k)^{-\frac{1}{2}}$), using a result from Nourdin and Peccati [4] in their recent book on the Stein method.

The original motivation for this paper was [2], where Baudoin and Nualart studied a complex-valued fBm with $H > 1/2$. For $B = B_t^1 + iB_t^2$, $B_0 = 1$, they studied the integral

$$\int_0^t \frac{dB_s}{B_s} = \int_0^t \frac{B_s^1 dB_s^1 + B_s^2 dB_s^2}{|B_s|^2} + i \int_0^t \frac{B_s^2 dB_s^1 - B_s^1 dB_s^2}{|B_s|^2}. \quad (1)$$

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When B is written in the form $\rho_t e^{i\theta_t}$, the angle θ_t is given by the imaginary part of (1). For standard Brownian motion, a well-known theorem by Spitzer [10] holds that as $t \rightarrow \infty$, the random variable $2\theta_t/(\log t)$ converges in distribution to a Cauchy random variable with parameter 1. We are not aware of a corresponding fBm version of Spitzer's theorem. In [2], the functional

$$Z_t := \int_1^t \frac{B_s^2 dB_s^1 - B_s^1 dB_s^2}{s^{2H}} \quad (2)$$

was proposed as an asymptotic approximation for θ_t . It was shown (see Proposition 22 of [2]) that $\frac{Z_t}{\sqrt{\log t}}$ converges in distribution to a Gaussian random variable, with an expression for variance similar to our own result. Their proof also used Malliavin calculus, but did not use the Fourth Moment Theorem. For $q = 2$, since $B_t = \int_0^t dB_s$, Z_t is asymptotically equal in law to

$$Z'_t = \int_1^t \int_1^s \frac{dB_r^2 dB_s^1}{s^{2H}} - \int_1^t \int_1^s \frac{dB_r^1 dB_s^2}{s^{2H}},$$

and we have a new (and shorter) proof of the result in [2].

The organization of this paper is as follows. In Section 2, we give necessary details about the theoretical background; this includes a brief discussion of Malliavin calculus, and some remarks about integration with respect to fBm, which has been studied extensively elsewhere. The Fourth Moment Theorem is also given. In Section 3, we state and prove the main result, which is Theorem 3.2. The proof follows in Sections 3.1 through 3.3; then Section 3.4 discusses the rate of convergence. Section 4 contains a technical lemma which was put at the end due to length.

2 Notation and Theory

We use the symbol $\mathbf{1}_A$ for the indicator function of a set A . Given a real n -tuple $\mathbf{x} := (x_1, \dots, x_n)$, we can re-arrange the variables in increasing order. We denote this re-ordered n -tuple $(x_{(1)}, \dots, x_{(n)})$. For a stochastic process $X = \{X(a), a \in \mathcal{I}\}$, we will use the notation X_a and $X(a)$ interchangeably. The symbol C denotes a generic positive constant, which may change from line to line.

2.1 Elements of Malliavin calculus

Following is a brief description of some identities that will be used in the paper. The reader may refer to [5] for detailed coverage of this topic. Let $X = \{X(h), h \in \mathcal{H}\}$ be an *isonormal Gaussian process* on a probability space (Ω, \mathcal{F}, P) , and indexed by a Hilbert space \mathcal{H} . That is, X is a family of Gaussian random variables such that $\mathbb{E}[X(h)] = 0$ and $\mathbb{E}[X(h)X(g)] = \langle h, g \rangle_{\mathcal{H}}$ for all $h, g \in \mathcal{H}$.

For integers $q \geq 1$, let $\mathcal{H}^{\otimes q}$ denote the q^{th} tensor product of \mathcal{H} . We use $\mathcal{H}^{\odot q}$ to denote the symmetric tensor product. Given a function $f \in \mathcal{H}^{\otimes q}$, we define the symmetrization $\tilde{f} \in \mathcal{H}^{\odot q}$ as

$$\tilde{f}(x_1, \dots, x_q) = \frac{1}{q!} \sum_{\sigma} f(x_{\sigma(1)}, \dots, x_{\sigma(q)}), \quad (3)$$

where σ includes all permutations of $\{1, \dots, q\}$.

Let $\{e_n, n \geq 1\}$ be a complete orthonormal system in \mathcal{H} . For functions $f, g \in \mathcal{H}^{\odot q}$ and $p \in \{0, \dots, q\}$, we define the p^{th} -order contraction of f and g as that element of $\mathcal{H}^{\otimes 2(q-p)}$ given by

$$f \otimes_p g = \sum_{i_1, \dots, i_p=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_p} \rangle_{\mathcal{H}^{\otimes p}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_p} \rangle_{\mathcal{H}^{\otimes p}} \quad (4)$$

where $f \otimes_0 g = f \otimes g$ and $f \otimes_q g = \langle f, g \rangle_{\mathcal{H}^{\otimes q}}$. While f and g are both symmetric, the contraction may not be. We denote its symmetrization by $f \tilde{\otimes}_p g$.

Let \mathcal{H}_q be the q^{th} Wiener chaos of X , that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where $H_q(x)$ is the q^{th} Hermite polynomial, defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}.$$

For $q \geq 1$, it is known that the map

$$I_q(h^{\otimes q}) = H_q(X(h)) \quad (5)$$

provides a linear isometry between the symmetric product space $\mathcal{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!} \|\cdot\|_{\mathcal{H}^{\otimes q}}$) and \mathcal{H}_q , where $I_q(\cdot)$ is the Wiener-Itô stochastic integral. By convention, $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$. It follows from (5) and the properties of the Hermite polynomials that for $f, g \in \mathcal{H}^{\odot q}$ we have

$$\mathbb{E}[I_q(f)I_q(g)] = q! \langle f, g \rangle_{\mathcal{H}^{\otimes q}}. \quad (6)$$

Let \mathcal{S} be the set of all smooth and cylindrical random variables of the form $F = g(X(\phi_1), \dots, X(\phi_n))$, where $n \geq 1$; $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is an infinitely differentiable function with compact support, and $\phi_i \in \mathcal{H}$. The Malliavin derivative of F with respect to X is the element of $L^2(\Omega, \mathcal{H})$ defined as

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(X(\phi_1), \dots, X(\phi_n)) \phi_i.$$

By iteration, for any integer $q > 1$ we can define the q^{th} derivative $D^q F$, which is an element of $L^2(\Omega, \mathcal{H}^{\odot q})$.

We let $\mathbb{D}^{q,2}$ denote the closure of \mathcal{S} with respect to the norm $\|\cdot\|_{\mathbb{D}^{q,2}}$ defined as

$$\|F\|_{\mathbb{D}^{q,2}}^2 = \mathbb{E}[F^2] + \sum_{i=1}^q \mathbb{E}[\|D^i F\|_{\mathcal{H}^{\otimes i}}^2].$$

We denote by δ the Skorohod integral, which is defined as the adjoint of the operator D . A random element $u \in L^2(\Omega, \mathcal{H})$ belongs to the domain of δ , $\text{Dom } \delta$, if and only if,

$$|\mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}]| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant which depends only on u . If $u \in \text{Dom } \delta$, then the random variable $\delta(u) \in L^2(\Omega)$ is defined for all $F \in \mathbb{D}^{1,2}$ by the duality relationship,

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}].$$

This is sometimes called the Malliavin integration by parts formula. We iteratively define the multiple Skorohod integral for $q \geq 1$ as $\delta(\delta^{q-1}(u))$, with $\delta^0(u) = u$. For this definition we have,

$$\mathbb{E}[F\delta^q(u)] = \mathbb{E}[\langle D^q F, u \rangle_{\mathcal{H}^{\otimes q}}],$$

where $u \in \text{Dom } \delta^q$ and $F \in \mathbb{D}^{q,2}$. The adjoint operator δ^q is an integral in the sense that for a (non-random) $h \in \mathcal{H}^{\odot q}$, we have $\delta^q(h) = I_q(h)$.

We will use the following hypercontractivity property of iterated integrals (see [7], Theorem 2.7.2, or [5], Sec. 1.4.3 for complete details). Let $f \in \mathcal{H}^{\odot q}$ and $p \geq 2$. Then there exists a positive constant $C_{p,q} < \infty$, depending only on p and q , such that

$$\mathbb{E}[|I_q(f)|^p] \leq C_{p,q} (\mathbb{E}[I_q(f)^2])^{\frac{p}{2}}. \quad (7)$$

2.2 Fractional Brownian motion

Fix $T > 0$ and an integer $d \geq 1$. Let $B = \{B_t, 0 \leq t \leq T\} = (B_t^1, \dots, B_t^d)$ be a d -dimensional fBm, that is, each B_t^i is an independent, centered Gaussian process with $B_0^i = 0$ and covariance

$$\mathbb{E}[B_s^i B_t^i] := R(s, t) = \frac{1}{2} (s^{2H} + t^{2H} - |s - t|^{2H})$$

for $t, s \geq 0$. We assume that $\frac{1}{2} < H < 1$. We will use the following elementary properties of $R(s, t)$:

(R.1) $R(s, t) = R(t, s)$; and for any $\epsilon > 0$, $R(s + \epsilon, t) \geq R(s, t)$.

(R.2) There are constants $1 \leq c_0 < c_1 \leq 2$ such that $c_0(st)^H \leq R(s, t) \leq c_1(st)^H$.

(R.3) As an alternate bound, if $s \leq t$ then the Mean Value Theorem implies

$$R(s, t) \leq s^{2H} + t^{2H} - (t - s)^{2H} \leq s^{2H} + st^{2H-1}.$$

Let \mathcal{E} denote the set of \mathbb{R} -valued step functions on $[0, T] \times \{1, \dots, d\}$. Note that any $f = f(t, i) \in \mathcal{E}$ may be written as a linear combination of elementary functions $e_t^k = \mathbf{1}_{[0, t] \times \{k\}}$. Let \mathfrak{H}_d be the Hilbert space defined as the closure of \mathcal{E} with respect to the inner product

$$\langle e_s^k, e_t^j \rangle_{\mathfrak{H}_d} = \mathbb{E}[B_s^k B_t^j] = R(s, t) \delta_{kj},$$

where δ_{kj} is the Kronecker delta. The mapping $e_t^k \mapsto B^k(t)$ can be extended to a linear isometry between \mathfrak{H}_d and the Gaussian space spanned by B . In this way, $\{B(h), h \in \mathfrak{H}_d\}$ is an isonormal Gaussian process as in Section 2.1.

Let $\alpha_H = H(2H - 1)$. It is well known that we can write

$$R(s, t) = \alpha_H \int_0^s \int_0^t |\eta - \theta|^{2H-2} d\eta d\theta. \quad (8)$$

Consequently, for $f, g \in \mathcal{E}$ we can write

$$\langle f, g \rangle_{\mathfrak{H}_d} = \alpha_H \sum_{i=1}^d \int_0^T \int_0^T f(s, i) g(t, i) |t - s|^{2H-2} ds dt. \quad (9)$$

We recall (see [5], Sec. 5.1.3) that \mathfrak{H}_d contains the linear subspace of measurable, \mathbb{R} -valued functions φ on $[0, T] \times \{1, \dots, d\}$ such that

$$\sum_{i=1}^d \int_0^T \int_0^T |\varphi(s, i)| |\varphi(t, i)| |t - s|^{2H-2} ds dt < \infty.$$

We denote this space $|\mathfrak{H}_d|$. Let $|\mathfrak{H}_d^{q,s}|$ be the space of symmetric functions $f : ([0, T] \times \{1, \dots, d\})^q \rightarrow \mathbb{R}$ such that

$$\sum_{i_1, \dots, i_q=1}^d \int_{[0, T]^{2q}} |f((\eta_1, i_1), \dots, (\eta_q, i_q))| |f((\theta_1, i_1), \dots, (\theta_q, i_q))| |\eta - \theta|^{2H-2} d\eta d\theta < \infty.$$

Then $|\mathfrak{H}_d^{q,s}| \subset \mathfrak{H}_d^{\odot q}$, and for $f, g \in |\mathfrak{H}_d^{q,s}|$ we can write (4) as

$$f \otimes_p g = \alpha_H^p \sum_{k=1}^d \int_{[0, T]^{2p}} f((\eta, k), (\mathbf{t}_1, \mathbf{i}_1)) g((\theta, k), (\mathbf{t}_2, \mathbf{i}_2)) \prod_{j=1}^p |\eta_j - \theta_j|^{2H-2} d\eta d\theta, \quad (10)$$

where

$$(\eta, k) = (\eta_1, k), \dots, (\eta_p, k); (\theta, k) = (\theta_1, k), \dots, (\theta_p, k); (\mathbf{t}_1, \mathbf{i}_1) = (t_1, i_1), \dots, (t_{q-p}, i_{q-p}); \text{ and } (\mathbf{t}_2, \mathbf{i}_2) = (t_{q-p+1}, i_{q-p+1}), \dots, (t_{2(q-p)}, i_{2(q-p)}).$$

2.3 Stochastic integration with respect to fBm

Let $F = g(B(\phi_1), \dots, B(\phi_n))$, where $n \geq 1$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$, $\phi_i \in \mathfrak{H}_d$, and g is a smooth function as in Section 2.1. The Malliavin derivative of F is an element of \mathfrak{H}_d (which is isomorphic to the product space $(\mathfrak{H}_1)^d$), and we can write $D = (D^{(1)}, \dots, D^{(d)})$, where

$$D_t^{(i)} F = \sum_{j=1}^n \frac{\partial g}{\partial x_j} (B(\phi_1), \dots, B(\phi_n)) \phi_j(t, i),$$

where we use the notation $D_t^{(i)} F = D^{(i)} F(t)$. We define the ‘component integral’ $\delta^{(i)}$ as the adjoint of $D^{(i)}$, and use the notation

$$\begin{aligned} \delta^{(i)}(u) &= \int_0^T u_t \delta B_t^i; \quad \text{and} \\ \delta(u) &= \int_0^T u_t \delta B_t = \sum_{i=1}^d \delta^{(i)}(u). \end{aligned} \tag{11}$$

where $u \in \text{Dom } \delta^{(i)} \subset L^2(\Omega, \mathfrak{H}_1)$ for every $i = 1, \dots, d$ implies $u \in \text{Dom } \delta \subset L^2(\Omega, \mathfrak{H}_d)$.

The pathwise stochastic integral with respect to fBm with $H > 1/2$ has been studied extensively [1, 3, 5]. For our purposes, we will use the symmetric Stratonovich integral discussed by Russo and Vallois [9]:

Definition 2.1. For some $T > 0$, let $u = \{u_t, 0 \leq t \leq T\}$ be a stochastic process with integrable trajectories. The symmetric integral with respect to the fBm B is defined as

$$\int_0^t u_s dB_s = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t u_s (B_{(s+\varepsilon) \wedge t} - B_{(s-\varepsilon) \vee 0}) ds,$$

where the limit exists in probability.

This theorem was first proved in [1].

Theorem 2.2. Let $u = \{u_t, t \geq 0\}$ be a stochastic process in $\mathbb{D}^{1,2}(\mathfrak{H}_1)$ such that, for some $T > 0$,

$$\begin{aligned} \mathbb{E} \left[\int_0^T \int_0^T |u_t| |u_s| |t-s|^{2H-2} ds dt \right] &< \infty; \\ \mathbb{E} \left[\int_{[0,T]^4} |D_t u_\theta| |D_s u_\eta| |t-s|^{2H-2} |\theta-\eta|^{2H-2} du dt d\theta d\eta \right] &< \infty; \\ \text{and } \int_0^T \int_0^T |D_s u_t| |t-s|^{2H-2} ds dt &< \infty \text{ a.s.} \end{aligned}$$

Then the limit of definition 2.1 exists in probability, and we have

$$\int_0^T u_t dB_t = \int_0^T u_t \delta B_t + \alpha_H \int_0^T \int_0^T D_s u_t |t-s|^{2H-2} ds dt,$$

where $\alpha_H = H(2H-1)$.

2.4 The Fourth Moment Theorem

Theorem 2.3. Fix integers $n \geq 2$ and $d \geq 1$. Let $\left\{ \left(f_1^{(k)}, \dots, f_d^{(k)} \right), k \geq 1 \right\}$ be a sequence of vectors such that $f_i^{(k)} \in \mathcal{H}^{\odot n}$ for each k and $i = 1, \dots, d$; and

$$\begin{aligned} \lim_{k \rightarrow \infty} n! \|f_i^{(k)}\|_{\mathcal{H}^{\otimes n}}^2 &= \lim_{k \rightarrow \infty} \left\| I_n \left(f_i^{(k)} \right) \right\|_{L^2(\Omega)}^2 = C_{ii}, \quad \forall i = 1, \dots, d; \\ \lim_{k \rightarrow \infty} \mathbb{E} \left[I_n \left(f_i^{(k)} \right) I_n \left(f_j^{(k)} \right) \right] &= C_{ij}, \quad \forall 1 \leq i < j \leq d. \end{aligned}$$

Then the following are equivalent:

- (i) As $k \rightarrow \infty$, the vector $\left(I_n(f_1^{(k)}), \dots, I_n(f_d^{(k)}) \right)$ converges in distribution to a d -dimensional Gaussian vector with distribution $\mathcal{N}(0, \mathbf{C}_d)$;
- (ii) For each $i = 1, \dots, d$, $I_n(f_i^{(k)})$ converges in distribution to N_i , where N_i is a centered Gaussian random variable with variance C_{ii} ;
- (iii) For each $i = 1, \dots, d$,

$$\lim_{k \rightarrow \infty} \mathbb{E} \left[I_n \left(f_i^{(k)} \right)^4 \right] = 3C_{ii}^2;$$

- (iv) For each $i = 1, \dots, d$, and each integer $1 \leq p \leq n-1$, $\lim_{k \rightarrow \infty} \left\| f_i^{(k)} \otimes_p f_i^{(k)} \right\|_{\mathcal{H}^{\otimes 2(n-p)}} = 0$.

This first version (which was 1-dimensional) of this theorem was proved in [7]. Since then, other equivalent conditions have been added [4, 6]. The multi-dimensional version stated above was proved by Peccati and Tudor [8]. A key advantage of this theorem is that, unlike the standard method of moments, it is not necessary to know about moments of any order higher than four.

3 Main result

Fix $q \geq 2$. For $t > 0$ and integer $k \geq 2$, define

$$Y_{k^t} = \int_1^{k^t} \int_1^{s_q} \dots \int_1^{s_2} s_q^{-qH} dB_{s_1}^1 \dots dB_{s_{q-1}}^{q-1} dB_{s_q}^q,$$

where the stochastic integrals are iterated symmetric integrals in the sense of Definition 2.1. Theorem 2.2 and the diagonal structure of Y_{k^t} allow us to identify the pathwise and Skorohod integrals.

Lemma 3.1. For each $q \geq 2$, we have

$$Y_{k^t} = \int_1^{k^t} \int_1^{s_q} \dots \int_1^{s_2} s_q^{-qH} \delta B_{s_1}^1 \dots \delta B_{s_{q-1}}^{q-1} \delta B_{s_q}^q. \quad (12)$$

Proof. This follows from iterated application of Theorem 2.2, where the correction term is zero due to independence. Indeed, in the notation of (11), this is

$$Y_{k^t} = \delta^{(q)} \dots \delta^{(1)} \left(s_q^{-qH} \mathbf{1}_{\{1 \leq s_1 < \dots < s_q \leq k^t\}} \right).$$

□

Following is the main result of this section.

Theorem 3.2. For $t \geq 0$, define

$$X_k(0) = 0; \quad X_k(t) = \frac{Y_{k^t}}{\sqrt{\log k}}, \quad t > 0.$$

Then as $k \rightarrow \infty$, the family $\{X_k(t), t \geq 0\}$ converges in distribution to the process $X = \{X(t), t \geq 0\}$, where X is a scaled Brownian motion with variance $\sigma_q^2 t$, and

$$\sigma_2^2 = \alpha_H \int_0^1 x^{-2H} R(1, x)(1-x)^{2H-2} dx; \quad \text{and for } q > 2, \quad (13)$$

$$\sigma_q^2 = \alpha_H^{q-1} \int_0^1 x_q^{-qH} (1-x_q)^{2H-2} \int_{\mathcal{M}} R(x_2, y_2) \prod_{i=2}^{q-1} |x_i - y_i|^{2H-2} dx_2 dy_2 \dots dy_{q-1} dx_q, \quad (14)$$

$$\text{where } \mathcal{M} = \{0 \leq x_2 < \dots < x_q; 0 \leq y_2 < \dots < y_{q-1} \leq 1\}.$$

The proof of Theorem 3.2 follows the lemmas in Sections 3.1 and 3.2. Our first task is to investigate the covariance (Section 3.1), then verify two other conditions for weak convergence (Section 3.2).

3.1 Convergence of the covariance function

Let $A = \{1 \leq s_1 < \dots < s_q \leq k^t\}$, and $B = \{(i_1, \dots, i_q) = (1, \dots, q)\}$. Lemma 3.1 allows us to write $Y_{k^t} = \delta^q(f_{k^t})$, where $f_{k^t} : ([0, \infty) \times \{1, \dots, q\})^q \rightarrow \mathbb{R}$ is given by

$$f_{k^t}((s_1, i_1), \dots, (s_q, i_q)) = s_q^{-qH} \mathbf{1}_A(s_1, \dots, s_q) \mathbf{1}_B(i_1, \dots, i_q). \quad (15)$$

Here, $f_{k^t} \in \mathfrak{H}^{\otimes q}$, where $\mathfrak{H} := \mathfrak{H}_q$ is the Hilbert space associated with a q -dimensional fBm (see Section 2.2). Clearly, f_{k^t} is not symmetric. Instead, we will work with the symmetrization defined in (3):

$$\tilde{f}_{k^t} = \frac{1}{q!} \sum_{\sigma} s_{\sigma(q)}^{-qH} \mathbf{1}_A(s_{\sigma(1)}, \dots, s_{\sigma(q)}) \mathbf{1}_B(i_{\sigma(1)}, \dots, i_{\sigma(q)}), \quad (16)$$

where σ covers all permutations of $\{1, \dots, q\}$. This gives equivalent results, by the relation $I_q(\tilde{f}) = I_q(f)$ (see [5], Sec. 1.1.2).

By definition \tilde{f}_{k^t} is nonzero only if $1 \leq s_{\sigma(1)} < \dots < s_{\sigma(q)} \leq k^t$ and $(i_{\sigma(1)}, \dots, i_{\sigma(q)}) = (1, \dots, q)$, hence it is possible to express \tilde{f}_{k^t} without a sum. Let σ be an arbitrary permutation of $\{1, \dots, q\}$, and let $A_{\sigma} = \{1 \leq s_{\sigma(1)} < \dots < s_{\sigma(q)} \leq k^t\}$. Since the sets $\{A_{\sigma}\}$ form an almost-everywhere partition of $[1, k^t]^q$, we can write (16) as

$$\tilde{f}_{k^t} = \frac{1}{q!} s_{(q)}^{-qH} \mathbf{1}_{A_1}((s_1, i_1), \dots, (s_q, i_q)), \quad (17)$$

where $s_{(q)} = \max\{s_1, \dots, s_q\}$, and the set A_1 is defined by the following condition: when s_1, \dots, s_q are arranged in $[1, k^t]$ such that $s_{(1)} < \dots < s_{(q)}$, then $(i_{(1)}, \dots, i_{(q)}) = (1, \dots, q)$.

In the next three results, we check the conditions of Theorem 2.3 for $\delta^q(\tilde{f}_{k^t})$.

Lemma 3.3. For each $q \geq 2$ and $t > 0$,

$$t\sigma_q^2 = \lim_{k \rightarrow \infty} \mathbb{E}[X_k(t)^2]$$

exists, where σ_q^2 is given by (13) and (14) for $q = 2$ and $q > 2$, respectively.

Proof. Since f_{k^t} is deterministic, we use (6) and (9):

$$\begin{aligned}\mathbb{E}[X_k(t)^2] &= \frac{1}{\log k} \mathbb{E}[\delta^q(f_{k^t})^2] = \frac{q!}{\log k} \left\langle \tilde{f}_{k^t}, \tilde{f}_{k^t} \right\rangle_{\mathfrak{H}^{\otimes q}} \\ &= \frac{\alpha_H^q}{q! \log k} \int_{[1, k^t]^{2q}} (r_{(q)} s_{(q)})^{-qH} \mathbf{1}_{A_1}(\mathbf{r}, \mathbf{i}) \mathbf{1}_{A_1}(\mathbf{s}, \mathbf{j}) \prod_{\ell=1}^q |r_\ell - s_\ell|^{2H-2} d\mathbf{r} d\mathbf{s},\end{aligned}\quad (18)$$

where $(\mathbf{r}, \mathbf{i}) = ((r_1, i_1), \dots, (r_q, i_q))$, and similar for (\mathbf{s}, \mathbf{j}) . To evaluate (18), we decompose $[1, k^t]^{2q}$ into the union of the sets $\{A_\sigma \times A_{\sigma'}\}$, which form a partition almost everywhere. Since $\mathbf{1}_{A_1}(\mathbf{r}, \mathbf{i})$ is nonzero only if $r_{\sigma(1)} < \dots < r_{\sigma(q)}$ and $(i_{\sigma(1)}, \dots, i_{\sigma(q)}) = (1, \dots, q)$, and similar for $\mathbf{1}_{A_1}(\mathbf{s}, \mathbf{j})$, it follows that we integrate only over the diagonal sets, that is, when $\sigma = \sigma'$. Hence, (18) can be integrated as a sum of $q!$ equal terms, and we have

$$\mathbb{E}[X_k(t)^2] = \frac{\alpha_H^q}{\log k} \int_{\mathcal{A}} (r_q s_q)^{-qH} \prod_{i=1}^q |r_i - s_i|^{2H-2} dr_1 ds_1 \dots dr_q ds_q, \quad (19)$$

where the integral is over the set

$$\mathcal{A} = \{1 \leq r_1 < \dots < r_q \leq k^t, 1 \leq s_1 < \dots < s_q \leq k^t\}.$$

Integrating over r_1, s_1 , we have by L'Hôpital,

$$\begin{aligned}\lim_{k \rightarrow \infty} \frac{\alpha_H^{q-1}}{\log k} \int_{[1, k^t]^2} (r_q s_q)^{-qH} |r_q - s_q|^{2H-2} \int_{\mathcal{A}'} R(r_2, s_2) \prod_{i=2}^{q-1} |r_i - s_i|^{2H-2} dr_2 ds_2 \dots dr_q ds_q \\ = \lim_{k \rightarrow \infty} t k^t \alpha_H^{q-1} \int_1^{k^t} (r_q k^t)^{-qH} (k^t - r_q)^{2H-2} \int_{\mathcal{A}'} R(r_2, s_2) \prod_{i=2}^{q-1} |r_i - s_i|^{2H-2} dr_2 ds_2 \dots ds_{q-1} dr_q,\end{aligned}$$

where the set $\mathcal{A}' = \{1 \leq r_2 < \dots < r_q, 1 \leq s_2 < \dots < s_{q-1} \leq k^t\}$ (\mathcal{A}' is empty if $q = 2$). Using the change of variable $r_i = k^t x_i$, $s_i = k^t y_i$, this may be written

$$\begin{aligned}\lim_{k \rightarrow \infty} t \alpha_H^{q-1} \int_{\frac{1}{k^t}}^1 x_q^{-qH} (1 - x_q)^{2H-2} \int_{\mathcal{M}} R(x_2, y_2) \prod_{i=2}^{q-1} |x_i - y_i|^{2H-2} dx_2 dy_2 \dots dy_{q-1} dx_q \\ = t \alpha_H^{q-1} \int_0^1 x_q^{-qH} (1 - x_q)^{2H-2} \int_{\mathcal{M}} R(x_2, y_2) \prod_{i=2}^{q-1} |x_i - y_i|^{2H-2} dx_2 dy_2 \dots dy_{q-1} dx_q,\end{aligned}\quad (20)$$

where \mathcal{M} is as in (14) for $q > 2$, and we have (13) if $q = 2$. To show (13) and (14) are convergent, we use properties (R.1) and (R.2), so that

$$\sigma_2^2 = \alpha_H \int_0^1 x^{-2H} (1 - x)^{2H-2} R(1, x) dx \leq c_1 \alpha_H \int_0^1 x^{-H} (1 - x)^{2H-2} dx < \infty$$

and for $q > 2$

$$\sigma_q^2 \leq \alpha_H \int_0^1 x_q^{-qH} (1 - x_q)^{2H-2} R(1, x_q)^{q-1} dx_q \leq c_1^{q-1} \alpha_H \int_0^1 x_q^{-H} (1 - x_q)^{2H-2} dx_q < \infty.$$

This concludes the proof. \square

Lemma 3.4. *Let $0 \leq \tau \leq t$. For each $q \geq 2$,*

$$\lim_{k \rightarrow \infty} \mathbb{E}[X_k(t)X_k(\tau)] = \sigma_q^2 \tau;$$

and consequently $\lim_{k \rightarrow \infty} \mathbb{E}[X(s)X(t)] = \sigma_q^2(s \wedge t)$ for all $0 \leq s, t < \infty$.

Proof.

$$\begin{aligned} \mathbb{E}[X_k(t)X_k(\tau)] &= \mathbb{E}[(X_k(t) - X_k(\tau) + X_k(\tau))X_k(\tau)] \\ &= \frac{1}{\log k} \mathbb{E}[(Y_{k^t} - Y_{k^\tau})Y_{k^\tau}] + \mathbb{E}[X_k(\tau)^2], \end{aligned}$$

where $\mathbb{E}[X_k(\tau)^2] \rightarrow \sigma_q^2 \tau$ by Lemma 3.3. Note that $Y_{k^t} - Y_{k^\tau} = \delta^q(\tilde{f}_{k^t}) - \delta^q(\tilde{f}_{k^\tau})$, where, recalling the notation of (17),

$$\begin{aligned} \delta^q(\tilde{f}_{k^t}) - \delta^q(\tilde{f}_{k^\tau}) &= \int_{[1, k^t]^{2q}} \frac{1}{q! s_{(q)}^{qH}} \mathbf{1}_{A_1}(\mathbf{s}, \mathbf{i}) \delta B_s - \int_{[1, k^\tau]^{2q}} \frac{1}{q! s_{(q)}^{qH}} \mathbf{1}_{A_1}(\mathbf{s}, \mathbf{i}) \delta B_s \\ &= \int_1^{k^t} \int_1^{s_{(q)}} \cdots \int_1^{s_{(2)}} \frac{1}{q! s_{(q)}^{qH}} \delta B_{s_{(1)}}^{(1)} \cdots \delta B_{s_{(q-1)}}^{(q-1)} \delta B_{s_{(q)}}^{(q)} - \int_1^{k^\tau} \int_1^{s_{(q)}} \cdots \int_1^{s_{(2)}} \frac{1}{q! s_{(q)}^{qH}} \delta B_{s_{(1)}}^{(1)} \cdots \delta B_{s_{(q-1)}}^{(q-1)} \delta B_{s_{(q)}}^{(q)} \\ &= \int_{k^\tau}^{k^t} \int_1^{s_{(q)}} \cdots \int_1^{s_{(2)}} \frac{1}{q! s_{(q)}^{qH}} \delta B_{s_{(1)}}^{(1)} \cdots \delta B_{s_{(q-1)}}^{(q-1)} \delta B_{s_{(q)}}^{(q)}. \end{aligned}$$

Hence, we can write $Y_{k^t} - Y_{k^\tau} = \delta^q(\tilde{f}_{\Delta k})$, where

$$\tilde{f}_{\Delta k} = \frac{1}{q! s_{(q)}^{qH}} \mathbf{1}_{A_1} \mathbf{1}_{\{k^\tau \leq s_{(q)} \leq k^t\}} = \tilde{f}_{k^t} \mathbf{1}_{\{k^\tau \leq s_{(q)} \leq k^t\}}. \quad (21)$$

With this notation, it follows that

$$\begin{aligned} \frac{1}{\log k} \mathbb{E}[(Y_{k^t} - Y_{k^\tau})Y_{k^\tau}] &= \frac{q!}{\log k} \left\langle \tilde{f}_{\Delta k}, \tilde{f}_{k^\tau} \right\rangle_{\mathfrak{H}^{\otimes q}} \\ &= \frac{\alpha_H^q}{q! \log k} \int_{[1, k^t]^{2q}} (r_{(q)} s_{(q)})^{-qH} \mathbf{1}_{A_1}(\mathbf{r}, \mathbf{i}) \mathbf{1}_{A_1}(\mathbf{s}, \mathbf{j}) \mathbf{1}_{\{1 \leq s_{(q)} \leq k^\tau \leq r_{(q)} \leq k^t\}} \prod_{\ell=1}^q |r_\ell - s_\ell|^{2H-2} ds \, d\mathbf{r}. \end{aligned}$$

As in Lemma 3.3, we decompose $[1, k^t]^{2q}$ into the union of the sets $\{A_\sigma \times A_{\sigma'}\}$. Since $\mathbf{1}_{A_1}(\mathbf{r}, \mathbf{i})$ is nonzero only if $r_{\sigma(1)} < \cdots < r_{\sigma(q)}$ and $(i_{\sigma(1)}, \dots, i_{\sigma(q)}) = (1, \dots, q)$, and similar for $\mathbf{1}_{A_1}(\mathbf{s}, \mathbf{j})$, it follows that we integrate only over the diagonal sets, that is, when $\sigma = \sigma'$. Hence, we have $q!$ equal terms of the form

$$\frac{\alpha_H^q}{q! \log k} \int_{k^\tau}^{k^t} \int_1^{k^\tau} \int_{[1, k^t]^{2q-2}} (r_q s_q)^{-qH} \mathbf{1}_{\{r_1 < \cdots < r_q\}} \mathbf{1}_{\{s_1 < \cdots < s_q\}} \prod_{\ell=1}^q |r_\ell - s_\ell|^{2H-2} ds \, d\mathbf{r}. \quad (22)$$

By (R.1) and (R.2), for each $r_\ell \leq r_q$, $s_\ell \leq s_q$, we have the estimate

$$\begin{aligned} &\alpha_H \int_1^{r^{(\ell)}} \int_1^{s^{(\ell)}} |r_{(\ell-1)} - s_{(\ell-1)}|^{2H-2} dr_{(\ell-1)} \, ds_{(\ell-1)} \\ &\leq \alpha_H \int_0^{r^{(q)}} \int_0^{s^{(q)}} |r - s|^{2H-2} dr \, ds = R(r_{(q)}, s_{(q)}) \leq c_1 (r_q s_q)^H. \end{aligned} \quad (23)$$

It follows that

$$\begin{aligned} \frac{1}{\log k} \mathbb{E} [(Y_{k^t} - Y_{k^\tau}) Y_{k^\tau}] &\leq \frac{C}{\log k} \int_{k^\tau}^{k^t} \int_1^{k^\tau} (r_q s_q)^{-qH} R(r_q, s_q)^{q-1} |r_q - s_q|^{2H-2} dr_q ds_q \\ &\leq \frac{C}{\log k} \int_{k^\tau}^{k^t} \int_1^{k^\tau} (r_q s_q)^{-2H} R(r_q, s_q) |r_q - s_q|^{2H-2} dr_q ds_q. \end{aligned}$$

Using the change-of-variable $s_q = k^\tau x$, $r_q = k^\tau y$, this is bounded by

$$\frac{C}{\log k} \int_1^{k^{t-\tau}} \int_0^1 (xy)^{-2H} R(x, y) (y - x)^{2H-2} dx dy.$$

Using (R.3), we obtain the estimate,

$$\frac{C}{\log k} \int_1^{k^{t-\tau}} \int_0^1 (y^{-2H} (y - x)^{2H-2} + x^{1-2H} y^{-1} (y - x)^{2H-2}) dx dy,$$

where

$$\begin{aligned} \int_1^{k^{t-\tau}} \int_0^1 y^{-2H} (y - x)^{2H-2} dx dy &\leq \int_1^2 y^{-2H} \int_0^y (y - x)^{2H-2} dx dy + \int_2^{k^{t-\tau}} (y - 1)^{-2} dy \\ &\leq C \int_1^2 y^{-1} dy + C \int_1^\infty y^{-2} dy < \infty, \end{aligned}$$

and

$$\begin{aligned} \int_1^{k^{t-\tau}} \int_0^1 y^{-1} x^{1-2H} (y - x)^{2H-2} dx dy &\leq \int_1^2 y^{-1} \int_0^y x^{1-2H} (y - x)^{2H-2} dx dy + \int_2^{k^{t-\tau}} (y - 1)^{2H-3} dy \\ &\leq C \int_1^2 y^{-1} dy + \int_1^\infty y^{2H-3} dy < \infty. \end{aligned}$$

Hence, this term vanishes and Lemma 3.4 is proved. \square

3.2 Conditions for weak convergence of $\{X_k(t)\}$

In the next two lemmas we verify additional properties of $\{X_k(t)\}$. In Lemma 3.5 we check condition (iv) of Theorem 2.3, and Lemma 3.6 is a tightness result.

Lemma 3.5. *Fix $q \geq 2$ and $t > 0$. For each integer $1 \leq p \leq q - 1$,*

$$\lim_{k \rightarrow \infty} (\log k)^{-2} \|\tilde{f}_{k^t} \otimes_p \tilde{f}_{k^t}\|_{\mathfrak{H}^{\otimes 2(q-p)}}^2 = 0.$$

Proof. Let $1 \leq p \leq q - 1$. To compute the p^{th} contraction of \tilde{f}_{k^t} , we use (10).

$$\tilde{f}_{k^t} \otimes_p \tilde{f}_{k^t} = \frac{\alpha_H^p}{(q!)^2} \int_{[1, k^t]^{2p}} (r_{(q)} s_{(q)})^{-qH} \mathbf{1}_{A_1}(\mathbf{r}, \mathbf{i}) \mathbf{1}_{A_1}(\mathbf{s}, \mathbf{j}) \prod_{\ell=1}^p |r_\ell - s_\ell|^{2H-2} dr_1 ds_1 \dots dr_p ds_p. \quad (24)$$

Using (24), we want to compute

$$\|\tilde{f}_{k^t} \otimes_p \tilde{f}_{k^t}\|_{\mathfrak{H}^{\otimes 2(q-p)}}^2 = \left\langle \tilde{f}_{k^t} \otimes_p \tilde{f}_{k^t}, \tilde{f}_{k^t} \otimes_p \tilde{f}_{k^t} \right\rangle_{\mathfrak{H}^{\otimes 2(p-q)}}$$

$$\begin{aligned}
&= \frac{\alpha_H^{2q}}{(q!)^4} \int_{[1, k^t]^{4q}} \left(r_{(q)} s_{(q)} r'_{(q)} s'_{(q)} \right)^{-qH} (\mathbf{1}_{A_1})^4 \prod_{i=1}^p (|r_i - s_i| |r'_i - s'_i|)^{2H-2} \\
&\quad \times \prod_{i=p+1}^q (|r_i - r'_i| |s_i - s'_i|)^{2H-2} d\mathbf{r} ds d\mathbf{r}' ds'. \quad (25)
\end{aligned}$$

As in the proof of Lemma 3.3, we view integration over the set $[1, k^t]^{4q}$ as a sum of integrals over various cases corresponding to the orderings of the real variables r_1, \dots, r_q (as in Lemma 3.3, the variables $\mathbf{s}, \mathbf{r}', \mathbf{s}'$ must follow the same ordering). Up to permutation of indices, each integral term has the form

$$\frac{\alpha_H^{2q}}{(q!)^4} \int_{\mathcal{G}} (r_{(q)} s_{(q)} r'_{(q)} s'_{(q)})^{-qH} \prod_{i=1}^p (|r_i - s_i| |r'_i - s'_i|)^{2H-2} \prod_{i=p+1}^q (|r_i - r'_i| |s_i - s'_i|)^{2H-2} d\mathbf{r} ds d\mathbf{r}' ds', \quad (26)$$

where $\mathcal{G} = \{1 \leq r_{(1)} < \dots < r_{(q)} \leq k^t; \dots; 1 \leq s'_{(1)} < \dots < s'_{(q)} \leq k^t\}$. To evaluate (26), there are two cases to consider. The first case is if $r_{(q)} \in \{r_1, \dots, r_p\}$, that is, (26) contains the terms $|r_{(q)} - s_{(q)}|, |r'_{(q)} - s'_{(q)}|$. In this case, using (23) we can bound (26) by

$$\begin{aligned}
&\frac{\alpha_H^2}{(q!)^4} \int_{[1, k^t]^4} (r_{(q)} s_{(q)} r'_{(q)} s'_{(q)})^{-qH} \left(R(r_{(q)}, s_{(q)}) R(r'_{(q)}, s'_{(q)}) \right)^{p-1} \left(R(r_{(q)}, r'_{(q)}) R(s_{(q)}, s'_{(q)}) \right)^{q-p} \\
&\quad \times \left(|r_{(q)} - s_{(q)}| |r'_{(q)} - s'_{(q)}| \right)^{2H-2} dr_{(q)} ds_{(q)} dr'_{(q)} ds'_{(q)} \\
&\leq C \int_{[1, k^t]^4} (rsr's')^{-2H} R(r, r') R(s, s') (|r - s| |r' - s'|)^{2H-2} dr ds dr' ds', \quad (27)
\end{aligned}$$

where we used (R.2) in the last estimate. The second case is the complement, that is, $r_{(q)} \in \{r_{p+1}, \dots, r_q\}$, so that (26) contains the terms $|r_{(q)} - r'_{(q)}|, |s_{(q)} - s'_{(q)}|$. If this is the case, then (26) is bounded by

$$\begin{aligned}
&\frac{\alpha_H^2}{(q!)^4} \int_{[1, k^t]^4} (r_{(q)} s_{(q)} r'_{(q)} s'_{(q)})^{-qH} \left(R(r_{(q)}, s_{(q)}) R(r'_{(q)}, s'_{(q)}) \right)^p \left(R(r_{(q)}, r'_{(q)}) R(s_{(q)}, s'_{(q)}) \right)^{q-p-1} \\
&\quad \times \left(|r_{(q)} - s_{(q)}| |r'_{(q)} - s'_{(q)}| \right)^{2H-2} dr_{(q)} ds_{(q)} dr'_{(q)} ds'_{(q)} \\
&\leq C \int_{[1, k^t]^4} (rsr's')^{-2H} R(r, s) R(r', s') (|r - r'| |s - s'|)^{2H-2} dr ds dr' ds'. \quad (28)
\end{aligned}$$

The result then follows by a change of variable and applying Lemma 4.1 to (27) and (28). \square

Lemma 3.6. *There is a constant $0 < C < \infty$ such that for each $k \geq 2$ and any $0 \leq \tau < t < \infty$ we have*

$$\mathbb{E} [|X_k(t) - X_k(\tau)|^4] \leq C(t - \tau)^2.$$

Proof. Based on the hypercontractivity property (7), it is enough to show

$$\mathbb{E} [|X_k(t) - X_k(\tau)|^2] \leq C(t - \tau).$$

Using the notation of (21), we can write

$$\begin{aligned}
\mathbb{E} [|X_k(t) - X_k(\tau)|^2] &= \frac{1}{\log k} \mathbb{E} [|Y_{k^t} - Y_{k^\tau}|^2] = \frac{q!}{\log k} \left\langle \tilde{f}_{\Delta k}, \tilde{f}_{\Delta k} \right\rangle_{\mathfrak{H}^{\otimes q}} \\
&= \frac{\alpha_H^q}{q! \log k} \int_{[1, k^t]^{2q}} (r_{(q)} s_{(q)})^{-qH} \mathbf{1}_{A_1}(\mathbf{r}, \mathbf{i}) \mathbf{1}_{A_1}(\mathbf{s}, \mathbf{j}) \mathbf{1}_{\{k^\tau \leq r_{(q)}, s_{(q)} \leq k^t\}} \prod_{\ell=1}^q |r_\ell - s_\ell|^{2H-2} d\mathbf{s} d\mathbf{r}.
\end{aligned}$$

In the same manner as (22), this can be decomposed into a sum of $q!$ equal terms of the form

$$\frac{\alpha_H^q}{q! \log k} \int_{k^\tau}^{k^t} \int_{k^\tau}^{k^t} \int_{[1, k^t]^{2q-2}} (r_q s_q)^{-qH} \mathbf{1}_{\{r_1 \leq \dots \leq r_q\}} \mathbf{1}_{\{s_1 \leq \dots \leq s_q\}} \prod_{\ell=1}^q |r_\ell - s_\ell|^{2H-2} d\mathbf{s} \, d\mathbf{r}.$$

Similar to Lemma 3.5, we use (23) and a change-of-variable to obtain

$$\begin{aligned} \frac{1}{\log k} \mathbb{E} [|Y_{k^t} - Y_{k^\tau}|^2] &\leq \frac{C}{\log k} \int_{k^\tau}^{k^t} \int_{k^\tau}^{k^t} (r_q s_q)^{-qH} R(r_q, s_q)^{q-1} |r_q - s_q|^{2H-2} dr_q \, ds_q \\ &\leq \frac{C}{\log k} \int_{k^{\tau-t}}^1 \int_{k^{\tau-t}}^1 (xy)^{-2H} R(x, y) |x - y|^{2H-2} dx \, dy. \end{aligned}$$

Without loss of generality, assume $x < y$. By (R.3), we have the estimate

$$\begin{aligned} \frac{C}{\log k} \int_{k^{\tau-t}}^1 \int_{k^{\tau-t}}^1 (xy)^{-2H} R(x, y) |x - y|^{2H-2} dx \, dy &= \frac{C}{\log k} \int_{k^{\tau-t}}^1 \int_{k^{\tau-t}}^y (xy)^{-2H} R(x, y) |x - y|^{2H-2} dx \, dy \\ &\leq \frac{C}{\log k} \int_{k^{\tau-t}}^1 \int_0^y y^{-2H} (y - x)^{2H-2} + x^{1-2H} y^{-1} (y - x)^{2H-2} dx \, dy \\ &\leq \frac{C}{\log k} \int_{k^{\tau-t}}^1 y^{-2H} y^{2H-1} + y^{-1} \, dy \leq C(t - \tau). \end{aligned}$$

This concludes the proof. \square

3.3 Proof of Theorem 3.2

Fix integers $q \geq 2$ and $d \geq 1$, and choose a set of times $0 \leq t_1 < \dots < t_d$. Lemmas 3.3 and 3.4 show that the random vector sequence $\{(X_k(t_1), \dots, X_k(t_d)), k \geq 1\}$ meets the covariance conditions of Theorem 2.3. Moreover, Lemma 3.5 verifies condition (iv) of Theorem 2.3. Therefore, we conclude that as $k \rightarrow \infty$,

$$(X_k(t_1), \dots, X_k(t_d)) \xrightarrow{\mathcal{L}} (X(t_1), \dots, X(t_d)), \quad (29)$$

where each $X(t_i)$ has distribution $\mathcal{N}(0, \sigma_q^2 t_i)$, and $\mathbb{E}[X(t_i)X(t_k)] = \sigma_q^2(t_i \wedge t_k)$ for all $1 \leq i, k \leq d$. By Lemma 3.6, the sequence $\{X_k(t)\}$ is tight, hence it follows from (29) that the sequence converges in the sense of finite-dimensional distributions, and we conclude that the family $\{X_k(t), t \geq 0\}$ converges in distribution to the process $\{X(t), t \geq 0\} \stackrel{\mathcal{L}}{=} \{\sigma_q W_t, t \geq 0\}$, where W_t is a standard Brownian motion. This concludes the proof of Theorem 3.2.

3.4 Rate of convergence

Let $t > 0$ be fixed. By Theorem 3.2, it follows that the sequence $\{X_k(t), k \geq 1\}$ converges in distribution to a random variable $N(t)$, where $N(t) \sim \mathcal{N}(0, \sigma_q^2 t)$. Recent work by Nourdin and Peccati [4] has produced a stronger form of the Fourth Moment Theorem for the 1-dimensional case, that is, that the conditions of the Fourth Moment Theorem also imply convergence in the sense of total variation (as well as other metrics - see Theorem 5.2.6). The result below follows from Corollary 5.2.10 of [4].

Proposition 3.7. *Let $t \geq 0$. Then for sufficiently large k , there is a constant $0 < C < \infty$ such that*

$$d_{TV}(X_k(t), N(t)) \leq \frac{C}{\sqrt{\log k}},$$

where $d_{TV}(\cdot, \cdot)$ is total variation distance. Hence $X_k(t)$ converges as $k \rightarrow \infty$ to Gaussian in the sense of total variation.

Proof. The result follows from an estimate in [4] (Cor. 5.2.10):

$$d_{TV}(X_k(t), N(t)) \leq 2\sqrt{\frac{\mathbb{E}[X_k(t)^4] - 3\mathbb{E}[X_k(t)^2]^2}{3\mathbb{E}[X_k(t)^2]^2}} + \frac{2|\mathbb{E}[X_k(t)^2] - \sigma_q^2 t|}{\mathbb{E}[X_k(t)^2] \vee \sigma_q^2 t}. \quad (30)$$

To simplify notation, we will assume $t = 1$. To help interpret this estimate, the following identity is computed in [4] (see Lemma 5.2.4):

$$\mathbb{E}[X_k(1)^4] - 3\mathbb{E}[X_k(1)^2]^2 = \frac{3}{q(\log k)^2} \sum_{p=1}^{q-1} p(p!)^2 \binom{q}{p}^4 (2q-2p)! \|\tilde{f}_k \otimes_p \tilde{f}_k\|_{\mathfrak{H}^{\otimes 2(q-p)}}^2. \quad (31)$$

From Lemma 3.5, we know $(\log k)^{-2} \|\tilde{f}_k \otimes_p \tilde{f}_k\|_{\mathfrak{H}^{\otimes 2(q-p)}}^2 \rightarrow 0$ at a rate $C/\log k$, hence it follows the first term of (30) is of order $C(\log k)^{-\frac{1}{2}}$. The second term depends on the convergence rate of (19). In the proof of Lemma 3.3, convergence follows from a limit of the form $\mathbb{E}[Y_k^2]/\log k$. By L'Hôpital's rule, it follows the rate of convergence has the form $C/\log k$, hence the first term controls. \square

4 A technical lemma

Lemma 4.1. *Fix $T > 0$. Let $1/2 < H < 1$, and for nonnegative x, y , let $R(x, y) = \frac{1}{2}(x^{2H} + y^{2H} - |x - y|^{2H})$. Then there is a constant $0 < K < \infty$ such that*

$$\int_{[\frac{1}{T}, 1]^4} (xyuv)^{-2H} R(x, y) R(u, v) |x - u|^{2H-2} |y - v|^{2H-2} dx dy du dv \leq K \log T.$$

Proof. In the following computations, we will obtain estimates based on the order of integration. Due to the symmetries of the integral, it is enough to consider four distinct cases. We will make frequent use of (R.3), and for a second estimate, note that for $x < y < u$ we can write $(u - x)^{2H-2} \leq (u - y)^{-\alpha} (y - x)^{-\beta}$, where $\alpha, \beta > 0$ satisfy $\alpha + \beta = 2 - 2H$.

Case 1: $x \leq y \leq u \leq v$ We can write

$$\begin{aligned} & \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^v (uv)^{-2H} R(u, v) \int_{\frac{1}{T}}^u y^{-2H} (v - y)^{2H-2} \int_{\frac{1}{T}}^y x^{-2H} R(x, y) (u - x)^{2H-2} dx dy du dv \\ & \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^v (uv)^{-2H} R(u, v) \int_{\frac{1}{T}}^u y^{-2H} (v - y)^{2H-2} (u - y)^{-\alpha} \int_{\frac{1}{T}}^y (y - x)^{-\beta} + x^{1-2H} y^{2H-1} (y - x)^{-\beta} dx \dots dv \\ & \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^v (uv)^{-2H} R(u, v) (v - u)^{-\alpha} \int_{\frac{1}{T}}^u y^{1-2H-\beta} (u - y)^{-\beta-\alpha} dy du dv \\ & \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^v (uv)^{-2H} R(u, v) (v - u)^{-\alpha} u^{-\beta} du dv \\ & \leq C \int_{\frac{1}{T}}^1 v^{-2H} \int_{\frac{1}{T}}^v u^{-2H} (v - u)^{-\alpha} (u^{2H} + uv^{2H-1}) du dv \\ & \leq C \int_{\frac{1}{T}}^1 v^{-1} dv \leq K \log T. \end{aligned}$$

Case 2: $x < y < v < u$ For this case, we use constants $\alpha, \beta > 0$ such that $\alpha + \beta = 2H - 2$, and $\gamma, \delta > 0$ such that $\gamma + \delta = \alpha$.

$$\begin{aligned}
& \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^u (uv)^{-2H} R(u, v) \int_{\frac{1}{T}}^u y^{-2H} (v - y)^{2H-2} \int_{\frac{1}{T}}^y x^{-2H} R(x, y) (u - x)^{2H-2} dx dy dv du \\
& \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^u (uv)^{-2H} R(u, v) \int_{\frac{1}{T}}^u y^{1-2H-\beta} (v - y)^{2H-2} (u - y)^{-\alpha} dy dv du \\
& \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^u (uv)^{-2H} R(u, v) (u - v)^{-\gamma} \int_{\frac{1}{T}}^u y^{1-2H-\beta} (v - y)^{2H-2-\delta} dy dv du \\
& \leq C \int_{\frac{1}{T}}^1 u^{-2H} \int_{\frac{1}{T}}^u v^{-2H-\beta-\delta} (v^{2H} + vu^{2H-1}) (u - v)^{-\gamma} dv du \\
& \leq C \int_{\frac{1}{T}}^1 u^{-1} \leq K \log T.
\end{aligned}$$

Case 3: $x < u < y < v$

$$\begin{aligned}
& \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^v (yv)^{-2H} (v - y)^{2H-2} \int_{\frac{1}{T}}^y u^{-2H} R(u, v) \int_{\frac{1}{T}}^u x^{-2H} R(x, y) (u - x)^{2H-2} dx du dy dv \\
& \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^v (yv)^{-2H} (v - y)^{2H-2} \int_{\frac{1}{T}}^y u^{-2H} (u^{2H} + uv^{2H-1}) (u^{2H-1} + y^{2H-1}) du dy dv \\
& \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^v (yv)^{-2H} (v - y)^{2H-2} \int_{\frac{1}{T}}^y (u^{2H-1} + y^{2H-1} + v^{2H-1} + u^{1-2H} (vy)^{2H-1}) du dy dv \\
& \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^v (yv)^{-2H} (v - y)^{2H-2} (y^{2H} + yv^{2H-1}) dy dv \\
& \leq C \int_{\frac{1}{T}}^1 v^{-1} dv \leq K \log T.
\end{aligned}$$

Case 4: $x < v < u < y$

$$\begin{aligned}
& \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^y (uy)^{-2H} \int_{\frac{1}{T}}^u v^{-2H} R(u, v) (y - v)^{2H-2} \int_{\frac{1}{T}}^v x^{-2H} R(x, y) (u - x)^{2H-2} dx dv du dy \\
& \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^y (uy)^{-2H} \int_{\frac{1}{T}}^u v^{-2H} R(u, v) (y - v)^{2H-2} (u - v)^{-\alpha} \int_{\frac{1}{T}}^v x^{-2H} (x^{2H} + xy^{2H-1}) (v - x)^{-\beta} dx dv du dy \\
& \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^y (uy)^{-2H} (y - u)^{-\alpha} \int_{\frac{1}{T}}^u v^{-2H} (v^{2H} + vu^{2H-1}) (u - v)^{-\alpha-\beta} (v^{1-\beta} + v^{2-2H-\beta} y^{2H-1}) dv du dy \\
& \leq C \int_{\frac{1}{T}}^1 \int_{\frac{1}{T}}^y (uy)^{-2H} (y - u)^{-\alpha} (u^{2H-\beta} + y^{2H-1} u^{1-\beta}) du dy \\
& \leq C \int_{\frac{1}{T}}^1 y^{-2H} (y^{1-\alpha-\beta} + y^{2H-1}) dy \\
& \leq C \int_{\frac{1}{T}}^1 y^{-1} dy \leq K \log T.
\end{aligned}$$

□

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